University of Manitoba
Department of Mathematics

Graduate Comprehensive Examination in Algebra

Thursday, September 24, 2015

9:00 AM – 3:00 PM

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Instructions (Please read carefully):

- You have altogether 6 hours to complete the examination.

- Part A consists of 10 questions worth two marks each. Answer all questions in Part A on the question paper itself. Each of these questions can and should be answered in no more than five sentences.

- You have a choice of questions in each of Parts B and C. The questions in Part B are worth 5 marks each. Answer any 8 questions out of 12 in this part. The questions in Part C are worth 10 marks each. Answer any 4 questions out of 6 in this part.

- You may attempt as many questions as you like in Parts B and C; however, if you attempt more than the required number of questions, you must clearly indicate which answers you want us to mark. In the absence of any explicit indication, we will mark respectively the first 8 questions for Part B, and first 4 questions for Part C in the order of their appearance in your answer booklets.

- In order to pass this examination, you must obtain at least 75% overall.
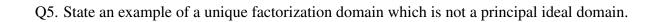
PART A

Please answer each of the following 10 questions in the space provided. Each correct answer is worth two marks. Each question should be answered briefly; i.e., in no more than five sentences.

Q1. Calculate the result of applying the Gram-Schmidt orthogonalization algorithm to

$$v_1 = \langle 1, 2, -2 \rangle; v_2 = \langle 2, 1, 3 \rangle.$$

Q2. Let V be the vector space of $\mathbb{C}^\infty$ functions on the real numbers $\mathbb{R}$, and D the linear operator of differentiation. Show that every real number is an eigenvalue of D, and state an eigenvector for each.
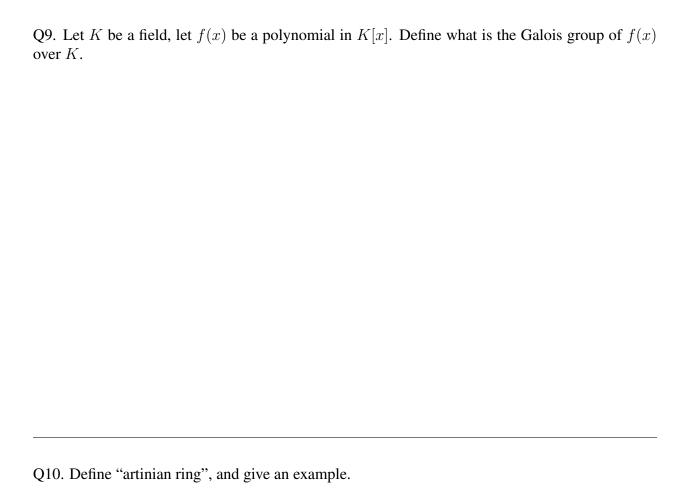
Q3. Let G be a group, and $x \in G$. Prove that any conjugate of x has the same order as x.

Q4. Define "solvable group".

Q5. State an example of a unique factorization domain which is not a principal ideal domain.

Q6. Let R be a ring. Prove that R is commutative if $x^2 = x$ for all $x \in R$.

Q7. Let R be a ring and P_R a right R module. Define "P is projective" and give an example.

Q8. Define what it means for a polynomial $f(x) \in k[x]$ (k a field) to be solvable by radicals.

Q9. Let K be a field, let $f(x)$ be a polynomial in $K[x]$. Define what is the Galois group of $f(x)$ over K.

Q10. Define "artinian ring", and give an example.

PART B

Please answer any 8 of the following 12 questions in your answer booklet. Each question is worth 5 marks. If you attempt more than 8, then please indicate clearly which ones you want us to mark.

Q1. Let $T : V \to V$ be a linear map with irreducible characteristic polynomial. Prove that if $\dim V = n$, then for every nonzero vector $v \in V$ the set $\{v, Tv, T^2v, \ldots, T^{n-1}v\}$ is a basis of V.

Q2. Prove that a finite dimensional vector space V and its dual V^* are isomorphic.

Q3. Prove that every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic.

Q4. Prove that as a group, the set of positive rationals under multiplication is isomorphic to a proper subgroup of itself.

Q5. Show that the ring $\mathbb{Z}[\sqrt{2}]$ has infinitely many units.

Q6. Let x be a nilpotent element in a commutative ring R with 1. Prove that $1 - ax$ is a unit of R for all $a \in R$.

Q7. By considering the degrees of field extensions $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt[6]{2})$ [i.e. the sixth root of 2], prove that the polynomial $x^3 - \sqrt{2}$ is irreducible over $\mathbb{Q}(\sqrt{2})$.

Q8. Let F be a finite field of characteristic p. Prove that the cardinality of F is p^n for some positive integer n.

Q9. Suppose that $F = \mathbb{Q}(\sqrt{a_1}, ..., \sqrt{a_n})$. Prove that $\sqrt[3]{2} \notin F$.

Q10. Let G be an abelian group. Show that

$$H = \{g | \exists n \text{ such that } g^n = 1\}$$

is a subgroup. Give an example showing that H is not a subgroup when G is nonabelian.

Q11. Let V be a vector space over a field k, and $T : V \to V$ be a linear map.
(a) Show that one can define a $k[x]$-module M by taking the underlying abelian group of V together with a $k[x]$-action where x acts on $v \in V$ by $x \cdot v = T(v)$.
(b) Show that there is a one-to-one correspondence between subspaces $U \subset V$ satisfying $T(U) = U$ and submodules of M.

Q12. Let R be a ring, and let
$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$
be an exact sequence of right R-modules and let N be a left R-module. Prove that

$$M' \otimes_R N \xrightarrow{f \otimes 1} M \otimes_R N \xrightarrow{g \otimes 1} M'' \otimes_R N \longrightarrow 0$$

is an exact sequence of abelian groups.

PART C

Please answer any 4 of the following 6 questions in your answer booklet. Each question is worth 10 marks. If you attempt more than 4, then please indicate clearly which ones you want us to mark.

Q1. Assume that $\{\vec{v}_1, \ldots, \vec{v}_m\}$ spans V and for some $k \leqslant m$ the first k of these vectors, $\vec{v}_1, \ldots, \vec{v}_k$, are linearly independent. Prove that a basis of V which contains $\{\vec{v}_1, \ldots, \vec{v}_k\}$ can be obtained by possibly discarding some vectors $\vec{v}_i$ for $i > k$.

Q2. Prove that any finite subgroup of the nonzero elements of a field is a cyclic group.

Q3. Prove that any group of order 200 has a normal Sylow-5 subgroup.

Q4. **(a)** Let R denote a principal ideal domain and $I \subset R$ a prime ideal. Show that R/I is a principal ideal domain.
(b) Show that if R is a principal ideal domain and $I \subset R$ is not prime, then R/I is not a unique factorization domain.
(c) If k is a field, prove that $k[x]$ is a principal ideal domain.
(d) Show that no quotient of $\mathbb{Q}[x]$ is isomorphic to $\mathbb{Z}[x]$.

Q5. Show that the quotient
$$\mathbb{F}_2[x]/(x^2 + x + 1)$$
is isomorphic to the set of matrices:
$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$
You may assume without proof that the given set of matrices forms a field under matrix multiplication and addition, when entries are taken to be elements of $\mathbb{F}_2$.

Q6. Let R be a commutative ring with 1 and M a right R-module.

a) Prove that the intersection of a descending chain of prime ideals of R is a prime ideal.

b) Prove that R has minimal prime ideals, and in fact every prime ideal contains a minimal prime ideal.